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Structural Parameters of Semi-Simple Lie Algebras

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Abstract

Based on the exploitations of properties of the Killing forms of semi-simple Lie algebras, we set out in a readily programmable form, the structural analysis and the Iwasawa-type decompositions of semi-simple Lie algebras. As an example, the case of $SO(3,1)$ and its covering group $SL(2, C)$ is worked out in some detail.

1. *Introduction*

With the constantly increasing involvement of Lie group theoretical concepts and methods in theoretical physics, particularly high energy physics, it has become more and more important for high energy physicists to go out of their way to gain some deeper understanding of the large body of mathematical information already available on abstract Lie algebras and Lie groups. In particular, one is often interested in the structure and representations of semi-simple Lie algebras.

One concept which has proved extremely useful in studying and characterising the structure of semi-simple Lie algebras is the Killing form of a Lie algebra. This concept leads for example to a neat solution of the problem of when a given group can be written as a product of elements of some of its subgroups. It turns out that by exploiting the properties of the Killing forms of Lie algebras, this problem can be solved first at the Lie algebra level, and thence by exponentiation, at the Lie group level.

In general, using the approach of Killing forms, one can set out the analysis of the structure of any semi-simple Lie algebra in a readily programmable form, with the result that the analysis of the structure of higher dimensional Lie algebras can easily be delegated to the computer. An as illustration of the procedure for such computations, we work out explicitly the structural parameters of the Lie algebra of $SO(3, 1)$ and its covering group $SL(2, C)$.

2. *Killing Forms of SO*(3, 1)

Let us consider, for example, the general pseudo-orthogonal group $SO(n-s, s)$, where s is the signature, and write down its Lie algebra which is given (Gourdin, 1967) by

$$[Z_{ij}, Z_{kl}] = g_{jk}Z_{il} - g_{ik}Z_{jl} + g_{il}Z_{jk} - g_{jl}Z_{ik} \tag{2.1}$$

with $Z_{ij} = -Z_{ji}$.

We may next specialise in the case of $SO(3, 1)$ for which we put

$$g = (+1, +1, +1, -1)$$

and choose our generators as

$$Z_{12}, Z_{31}, Z_{14}, Z_{23}, Z_{24}, Z_{34} \tag{2.2}$$

The result is the following set of commutation relations:

$$
\begin{aligned}
&[Z_{12}, Z_{31}] = +Z_{23}; \qquad (i=1, j=2, k=3, l=1 \text{ in equation (2.1)})\\
&[Z_{12}, Z_{14}] = -Z_{24}\\
&[Z_{12}, Z_{23}] = -Z_{31}\\
&[Z_{12}, Z_{24}] = +Z_{14}\\
&[Z_{12}, Z_{34}] = 0\\
&[Z_{31}, Z_{14}] = +Z_{34}\\
&[Z_{31}, Z_{23}] = +Z_{12}\\
&[Z_{31}, Z_{24}] = 0\\
&[Z_{31}, Z_{34}] = -Z_{14}\\
&[Z_{14}, Z_{23}] = 0\\
&[Z_{14}, Z_{24}] = +Z_{12}\\
&[Z_{14}, Z_{34}] = -Z_{31}\\
&[Z_{23}, Z_{24}] = -Z_{34}\\
&[Z_{23}, Z_{34}] = +Z_{24}\\
&[Z_{24}, Z_{34}] = +Z_{23}
\end{aligned} \tag{2.3}
$$

We may re-order these generators as follows:

$$
\begin{aligned}
&X_1 = Z_{12}; \qquad X_2 = Z_{31}; \qquad X_3 = Z_{24}\\
&X_4 = Z_{23}; \qquad X_5 = Z_{14}; \qquad X_6 = Z_{34}
\end{aligned}
$$

Then we obtain the following set of commutation relations and structure constants:

$$
\begin{array}{lll}
[X_1, X_2] = +X_4: & C_{124} = +1; & C_{214} = -1 \\
[X_1, X_3] = +X_5: & C_{135} = +1; & C_{315} = -1 \\
[X_1, X_4] = -X_2: & C_{142} = -1; & C_{412} = +1 \\
[X_1, X_5] = -X_3: & C_{153} = -1; & C_{513} = +1 \\
[X_1, X_6] = 0: & C_{16\tau} = C_{61\tau} = 0; & \tau = 1, 2 \ldots 6 \\
[X_2, X_3] = 0: & C_{23\tau} = C_{32\tau} = 0; & \tau = 1, 2 \ldots 6 \\
[X_2, X_4] = +X_1: & C_{241} = +1; & C_{421} = -1 \\
[X_2, X_5] = +X_6: & C_{256} = +1; & C_{526} = -1 \\
[X_2, X_6] = -X_5: & C_{265} = -1; & C_{625} = +1 \\
[X_3, X_4] = +X_6: & C_{346} = +1; & C_{436} = -1 \\
[X_3, X_5] = -X_1: & C_{351} = -1; & C_{531} = +1 \\
[X_3, X_6] = +X_4: & C_{364} = +1; & C_{634} = -1 \\
[X_4, X_5] = 0: & C_{45\tau} = C_{54\tau} = 0; & \tau = 1, 2 \ldots 6 \\
[X_4, X_6] = +X_3: & C_{463} = +1; & C_{643} = -1 \\
[X_5, X_6] = -X_2: & C_{562} = -1; & C_{652} = +1
\end{array}
\tag{2.4}
$$

The explicit 4 x 4 matrix representations of these generators may also be written down. It can be shown quite generally (Gourdin, 1967) that the Lie algebra of the general pseudo-orthogonal group $SO(n-s, s)$, is isomorphic to the matrix sub-algebra of $M(n, R)$, composed of $n \times n$ matrices of the form:

$$Z = \left(\begin{array}{c|c} Z_1 & Z_2 \\ \hline Z_2^T & Z_3 \end{array}\right)$$

where Z_1 is skew symmetric of order $(n-s)$, Z_3 is skew symmetric of order s, Z_2 is arbitrary $(n-s) \times s$ matrix, and Z_2^T is the transpose of Z_2.

Based on this, we can choose the following representations for the generators X_i of $SO(3, 1)$, the choice being made such that the commutation relations in equation (2.4) are preserved.

$$X_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad X_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \quad X_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}; \qquad X_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{2.5}$$

One checks that these matrices are indeed consistent with the commutation relations in equation (2.4).

Having now obtained the structure constants, we can compute the adjoint matrices associated with the generators.

In general we have

$$Ad_{X_i} = (C_i)_{jk}$$

which is an $n \times n$ matrix for an algebra of order n. Thus for $SO(3, 1)$ we have

$$Ad_{Xi} = \begin{pmatrix} C_{i11} & C_{i12} & C_{i13} & \dots & C_{i16} \\ C_{i21} & C_{i22} & \dots & \dots & \\ C_{i61} & C_{i62} & \dots & \dots & C_{i66} \end{pmatrix}$$

so that from the values of C_{ijk} given in equation (2.4) we obtain the following results:

$$Ad_{X_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Ad_{X_2} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

$$Ad_{X_3} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$Ad_{X_4} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$Ad_{X_5} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Ad_{X_6} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.6}$$

The Killing forms may next be deduced, defined generally by

$$B(X_i, X_j) = \mathrm{tr}\,(Ad_{X_i}\ Ad_{X_j})$$

For $SO(3, 1)$ we get the results:

$$\begin{aligned} B(X_i, X_i) &= -4 \qquad \text{for } i = 1, 2, 4 \\ &= +4 \qquad \text{for } i = 3, 5, 6 \end{aligned} \tag{2.7}$$

3. *Decompositions of* $SO(3, 1)$

Using now these explicit values of the Killing forms, as well as a number of theorems (Hermann, 1966; Helgason, 1962; Strom, 1971; Pontryagin, 1966; Nagel, 1969) dealing with the properties of Killing forms, the computation of the structural parameters of $SO(3, 1)$ and its Iwasawa decomposition, proceeds as follows.

We denote the set of all generators with negative Killing forms (or Killing norm) by L_K, while the remaining generators with positive Killing forms we denote by P. Thus for $SO(3, 1)$ we obtain:

$$\begin{aligned} L_K &= \{X_1, X_2, X_4\} \\ P &= \{X_3, X_5, X_6\} \end{aligned} \tag{3.1}$$

The generators X_1, X_2, X_4 with the negative-definite Killing forms, we call compact generators (Pontryagin, 1966), corresponding to the known fact that the subset L_K forms a compact subalgebra of L. It is in fact the maximal compact subalgebra of L. To find what this compact algebra is, we write down the commutation relations of the elements of L_K and try to establish isomorphsim with some known compact algebra (of the same order).

Thus from equations (3.1) and (4.1) we get:

$$\begin{aligned}[X_1, X_2] &= +X_4 \\ [X_1, X_4] &= -X_2 \\ [X_2, X_4] &= +X_1\end{aligned} \tag{3.2}$$

By using equation (2.1) and writing down directly, the Lie algebra of $SO(3, R)$, namely:

$$\begin{aligned}[Z_{23}, Z_{31}] &= -Z_{12} \\ [Z_{31}, Z_{12}] &= -Z_{23} \\ [Z_{12}, Z_{23}] &= -Z_{31}\end{aligned}$$

we establish the one-one correspondence:

$$Z_{12} \Leftrightarrow X_1; \qquad Z_{31} \Leftrightarrow X_2, \qquad Z_{23} \Leftrightarrow X_4 \tag{3.3}$$

so that $L_K = sO(3, R)$.

Next we use equations (2.4) and (3.2) to check the following properties of the subsets L_K and P. We have

$$\begin{aligned}[L_K, P] &\subset P \\ [P, P] &\subset L_K\end{aligned}$$

For example:

$$\begin{aligned}[X_1, X_3] &= +X_5 \in P \\ [X_1, X_6] &= 0 \in P \\ [X_2, X_6] &= -X_5 \in P \\ [X_3, X_5] &= -X_1 \in L_K \\ [X_5, X_6] &= -X_2 \in L_K\end{aligned} \tag{3.4}$$

These are merely consistency checks. Finally, with the computed values of the Killing forms we check also that

$$B(P, L_K) = B(L_K, P) = 0$$

meaning that the subspace P is orthogonal to the subspace L_K with respect to the Killing form. P is, however, not a subalgebra of L as seen from equation (3.4).

Algebraically, these results imply that we can write

$$\begin{aligned}sO(3, 1) &= L_K \oplus P \\ &= sO(3, R) \oplus P\end{aligned} \tag{3.5}$$

This gives us first-stage decomposition of $SO(3, 1)$.

4. *Decomposition of P*

With a view to further decomposition of equation (3.5) the prescription is to scan through P, and pick out all those elements X which are mutually commuting. The set of such elements we denote by L_A. The content of L_A is obtained from equation (2.4).

Thus for $P \in SO(3, 1)$ we have:

$$\begin{aligned}[X_3, X_5] &= -X_1 \\ [X_3, X_6] &= +X_4 \\ [X_5, X_6] &= -X_2\end{aligned} \tag{4.1}$$

from which we see that no two generators commute in this case, so that we may arbitrarily choose:

$$L_A = X_6 \tag{4.2}$$

In the general case, one denotes the number of elements of L_A by l. Thus:

$$L_A = \{H_i\}, \qquad i = 1, 2, \ldots l$$

Then we are required to set up eigenvalue equations for L_A as follows: We look for the set of all elements X belonging to the original algebra L, which satisfy:

$$[H, X] = \alpha(H)X \tag{4.3}$$

where for a given H, $\alpha(H)$ is a real number, known as the eigenvalue of H, while the element X is the eigenfunction of H. In general, there should be as many different eigenvalues as there are distinct eigenfunctions. Some eigenvalues may however be degenerate. The number of different eigenfunctions X having the same eigenvalue $\alpha(H)$, for a fixed H, we call the multiplicity or the degeneracy of the eigenvalue $\alpha(H)$. The set of all such degenerate eigenfunctions we denote by L^α. We need to compute these L^α quantities in order to gain more information about the structure of the algebra L. We therefore proceed further as follows:

5. *Spectrum of Eigenvalues and Roots*

In general for the given algebra L, a spectrum of eigenvalues may exist, and not just one eigenvalue. The first step in finding the set of L^α quantities, is to determine any eigenfunctions X which have eigenvalue zero. The set of such eigenfunctions we denote by L^0 corresponding to $\alpha(H) = 0$.

In the case of $SO(3, 1)$ we see from equations (4.3), (4.2) and (2.4) that

$$L^0 = \{X_1, X_6\} \tag{5.1}$$

The remaining non-zero eigenvalues are called roots, and are determined from equations (4.3), (4.2) and (2.4) as follows.

Having extracted L^0 from L, we take the remaining elements of L and write out their commutation relations with X_6 as follows:

$$[X_6, X_2] = +X_5$$
$$[X_6, X_3] = -X_4$$
$$[X_6, X_4] = -X_3$$
$$[X_6, X_5] = +X_2$$

We see that none of these equations is of the form (4.3), so that X_2, X_3, X_4 and X_5 are not eigenfunctions of X_6. We then try to form suitable linear combinations which will be eigenfunctions of X_6 (or in general of the H_i operators). Forming the quantities:

$$\begin{aligned} Z^- &= X_2 - X_5 \\ Z^+ &= X_2 + X_5 \\ W^- &= X_3 + X_4 \\ W^+ &= X_3 - X_4 \end{aligned} \tag{5.2}$$

we see that

$$\begin{aligned} [X_6, Z^+] &= Z^+ \\ [X_6, Z^-] &= -Z^- \\ [X_6, W^+] &= W^+ \\ [X_6, W^-] &= -W^- \end{aligned} \tag{5.3}$$

These are of the form (4.3), so that the eigenfunctions of X_6 are Z^+, W^+ each with eigenvalue $+1$; and Z^-, W^-, each with eigenvalue -1. We can now write

$$\begin{aligned} L^{(+1)} &= \{Z^+, W^+\} \\ L^{(-1)} &= \{Z^-, W^-\} \end{aligned} \tag{5.4}$$

The eigenvalue spectrum of $SO(3, 1)$ is then made up as follows:

$$\begin{aligned} &0 \text{ with multiplicity } 2 \\ &+1 \text{ with multiplicity } 2 \\ &-1 \text{ with multiplicity } 2 \end{aligned} \tag{5.5}$$

The roots of $SO(3, 1)$ are

$$+1 \text{ (twice)}$$
$$-1 \text{ (twice)}$$

We may write these four roots as

$$\begin{aligned} \alpha &= +1; \qquad -\alpha = -1 \\ \beta &= +1; \qquad -\beta = -1 \end{aligned} \tag{5.6}$$

We note that for the present case of $SO(3, 1)$ where L_A consists of just one operator X_6, the roots are points on the real line. In the more general case where L_A contains l operators, each root $\alpha, \beta, \ldots$ will be a vector in an l-dimensional dual space $\tilde{L}_A$. These roots are conventionally normalised by requiring that $\boldsymbol{\alpha}.\boldsymbol{\alpha} = \boldsymbol{\beta}.\boldsymbol{\beta} = \ldots = 1$. The roots in (5.6) are therefore already normalised.

6. *Normalisation of the Roots*

In the general case, the normalisation of the roots is stated in terms of the Killing forms of L as follows: We go back to equation (4.3) and consider the following two cases:

(i) $H = H_i$, where H_i is a basis vector in the l-dimensional Lie vector space L_A. Then equation (4.3) can be rewritten in the form:

$$[H_i, X] = \alpha(H_i)X = \alpha_i X \tag{6.1}$$

For $i = 1, 2, \ldots l$, and for a fixed eigenfunction X, we then get the quantities

$$(\alpha_1, \alpha_2, \ldots \alpha_l)$$

which may be considered as a vector $\boldsymbol{\alpha}$ existing in a separate l-dimensional space denoted by $\tilde{L}_A$, the dual space to the space L_A containing the operators H_i.

Now we may interpret equation (6.1) as follows. We can say that to every eigenvalue (or root) vector $\boldsymbol{\alpha}$ in space $\tilde{L}_A$, there is associated with it, a vector $\mathbf{H}_\alpha{}^1$ in space L_A, where $\mathbf{H}_\alpha{}^1$ has to be such that the Killing projection of $\mathbf{H}_\alpha{}^1$ on the basis vector H_i gives the number α_i. Thus given the set of numbers, $\{\alpha_i\}$ which are the root components in space $\tilde{L}_A$, we require to find that unique vector $\mathbf{H}_\alpha{}^1$ in space L_A, such that

$$B(H_i, \mathbf{H}_\alpha{}^1) = \alpha_i$$

or

$$\operatorname{tr}(Ad_{H_i}\ Ad_{H_\alpha{}^1}) = \alpha_i \tag{6.2}$$

With this specification, the required vector can easily be found by means of equations (2.5) and (2.7).

(ii) Next suppose we take the operator H in equation (4.3) to be an arbitrary vector operator $\mathbf{H}$ in L_A. We may still look for the eigenvalue of such a vector operator. Denoting this eigenvalue by the real number $\alpha(\mathbf{H})$, one can show that $\alpha(\mathbf{H})$ is the Killing form of the two vectors $\mathbf{H}_\alpha{}^1$ and $\mathbf{H}$. Thus let us write the eigenvalue equation:

$$[\mathbf{H}, X] = \alpha(\mathbf{H})X \tag{6.3}$$

and expand $\mathbf{H}$ in the form:

$$\mathbf{H} = \sum_{i=1}^{l} \lambda_i H_i$$

Then equation (6.3) becomes:

$$\sum_{i=1}^{l} \lambda_i [H_i, X] = \alpha(\mathbf{H})X$$

or using equation (6.1) we get:

$$\sum_{i=1}^{l} \lambda_i \alpha_i X = \alpha(\mathbf{H})X \tag{6.4}$$

whence

$$\alpha(\mathbf{H}) = \sum_{i=1}^{l} \lambda_i \alpha_i = \boldsymbol{\alpha}.\mathbf{H} \tag{6.5}$$

But from (6.2) and (6.4) we can also write

$$\alpha(\mathbf{H}) = \sum_{i=1}^{l} \lambda_i B(H_i, \mathbf{H}_\alpha{}^1) = B(\mathbf{H}, \mathbf{H}_\alpha{}^1) \tag{6.6}$$

which may be compared with equation (6.5).

Alternatively we argue as follows. We write

$$B(H_i, H_j) = g_{ij}$$

$$B(H_i, \mathbf{H}) = \sum_{j=1}^{l} \lambda_j B(H_i, H_j)$$

$$= \sum_{j=1}^{l} \lambda_j g_{ij}$$

$$B(H_i, \mathbf{H}^1) = \sum_{j=1}^{l} \lambda_j{}^1 g_{ij}$$

or

$$B(H_i, \mathbf{H}_\alpha{}^1) = \sum_{i=1}^{l} \lambda_j{}^1 g_{ij} = \alpha_i, \quad \text{say}$$

Then

$$B(\mathbf{H}, \mathbf{H}_\alpha{}^1) = \sum_{i=1}^{l} \lambda_i B(H_i, \mathbf{H}_\alpha{}^1)$$

$$= \sum_{i=1}^{l} \lambda_i \alpha_i$$

$$= \alpha(\mathbf{H})$$

from equation (6.5). Then $\alpha(\mathbf{H}) = B(\mathbf{H}, \mathbf{H}_\alpha{}^1)$ as before.

This proves that if X is an eigenfunction of the individual basis operators H_i, with eigenvalue α_i, then X is also an eigenfunction of any vector operator $\mathbf{H}$, and has eigenvalue $\alpha(\mathbf{H})$ which is the scalar product of $\boldsymbol{\alpha}$ and $\mathbf{H}$ in a joint l-dimensional space. Now the eigenvalue $\alpha(\mathbf{H})$ of the vector operator $\mathbf{H}$ is always normalized to 2. That is,

$$\alpha(\mathbf{H}) = B(\mathbf{H}, \mathbf{H}_\alpha{}^1) = 2$$

Given therefore the adjoint matrix for $\mathbf{H}_\alpha{}^1$ and the eigenvalue 2, we can look for the operator $\mathbf{H} = \mathbf{H}_\alpha$ say, which is such that

$$B(\mathbf{H}_\alpha, \mathbf{H}_\alpha{}^1) = 2 \tag{6.7}$$

This operator $\mathbf{H}_\alpha$ in space L_A we associate with the eigenvalue $\alpha(\mathbf{H}_\alpha) = 2$.

Summarising, we note that if the root vectors $\boldsymbol{\alpha}$ and the operators $\mathbf{H}$ were to be considered as existing in the same l-dimensional space, then α_i, which is the eigenvalue of H_i, is to be interpreted as the projection of some vector $\mathbf{H}_\alpha{}^1$ on the basis vector H_i. This vector $\mathbf{H}_\alpha{}^1$ will be in the same direction as $\boldsymbol{\alpha}$ and is in fact identical with it.

On the other hand, the eigenvalue $\alpha(\mathbf{H})$ of some arbitrary vector operator $\mathbf{H}$ is to be interpreted as the Killing form $B(\mathbf{H}, \mathbf{H}_\alpha{}^1)$, or the scalar product of $\mathbf{H}$ with the vector $\mathbf{H}_\alpha{}^1$. Once we know the set of numbers $\boldsymbol{\alpha} = (\alpha_1, \alpha_2 \ldots \alpha_l)$ we can determine $\mathbf{H}_\alpha{}^1$ uniquely. For another root $\boldsymbol{\beta} = (\beta_1, \beta_2 \ldots \beta_l)$, we get another uniquely determined vector $\mathbf{H}_\beta{}^1$. There will in general be as many different such vectors $\mathbf{H}_\alpha{}^1, \mathbf{H}_\beta{}^1, \mathbf{H}_\gamma{}^1 \ldots$ as there are distinct roots. If X is the eigenvector with the set of eigenvalues $\boldsymbol{\alpha} = (\alpha_1, \alpha_2 \ldots \alpha_l)$, then the eigenvalue $\alpha(\mathbf{H})$ of an arbitrary operator $\mathbf{H}$ for the *same* eigenvector X, will be given by the Killing form of $\mathbf{H}_\alpha$ with the previously determined unique vector $\mathbf{H}_\alpha{}^1$. Among the set of all arbitrary vectors, we can find only one to be denoted by $\mathbf{H}_\alpha$ which is such that $\alpha(\mathbf{H}) = \alpha(\mathbf{H}_\alpha) = B(\mathbf{H}_\alpha \mathbf{H}_\alpha{}^1) = 2$.

Next, taking another eigenvector Y, with eigenvalues $\boldsymbol{\beta} = (\beta_1, \beta_2 \ldots \beta_l)$ for the same set of basis vectors H_i, we get that the eigenvalue of some arbitrary operator $\mathbf{H}$ for the same eigenvector Y, is given by:

$$\beta(\mathbf{H}) = B(\mathbf{H}, \mathbf{H}_\beta{}^1)$$

When this arbitrary vector $\mathbf{H}$ is selected such that the number $\beta(\mathbf{H}) = 2$, we re-name the operator $\mathbf{H}_\beta$ and write

$$\beta(\mathbf{H}) = \beta(\mathbf{H}_\beta) = B(\mathbf{H}_\beta, \mathbf{H}_\beta{}^1) = 2 \tag{6.8}$$

What finally emerges in this general case, is that given the roots $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \ldots$ we have first to compute the unique vectors $\mathbf{H}_\alpha{}^1, \mathbf{H}_\beta{}^1 \ldots$ from equation (6.2). Next we compute the vectors $\mathbf{H}_\alpha, \mathbf{H}_\beta$ from equation (6.7). We then claim that working subsequently with only these vectors, is equivalent to working with normalised roots. The normalisation in fact implies that once we have found the $\mathbf{H}_\alpha{}^1, \mathbf{H}_\beta{}^1 \ldots$ operators from equation (6.2), we can deduce the operators $\mathbf{H}_\alpha, \mathbf{H}_\beta \ldots$ from the formula:

$$\mathbf{H}_\alpha = \frac{2\mathbf{H}_\alpha{}^1}{B(\mathbf{H}_\alpha{}^1, \mathbf{H}_\alpha{}^1)} \tag{6.9}$$

This requires that having found the $\mathbf{H}_\alpha{}^1$ operators, we compute their Killing forms and scale these operators $\mathbf{H}_\alpha{}^1$ to the operators $\mathbf{H}_\alpha$ as in equation (6.9).

For the roots themselves, this normalisation is equivalent to the conventional one of normalisation to unity. Thus

$$\boldsymbol{\alpha}.\boldsymbol{\alpha} = \boldsymbol{\beta}.\boldsymbol{\beta} = \ldots = 1 \tag{6.10}$$

as stated before.

Returning to the specific case of $SO(3, 1)$ we see that the roots are already normalised. As a check, we compute the operators $\mathbf{H}_\alpha{}^1, \mathbf{H}_\beta{}^1, \mathbf{H}_\alpha$ and $\mathbf{H}_\beta$, associated with these normalised roots. Since the roots are one-component quantities, we have

$$\boldsymbol{\alpha} = \alpha_1 = +1$$
$$\boldsymbol{\beta} = \beta_1 = +1$$

Also

$$H_i = H_1 = X_6$$

Then from equation (6.2) we require

$$B(X_6, \mathbf{H}_\alpha{}^1) = B(X_6, H^1_{+1}) = +1$$

But from the earlier equation (2.7) we have $B(X_6, X_6) = +4$. Hence we may take $H^1_{+1} = \frac{1}{4}X_6$. Then

$$B(\mathbf{H}_\alpha{}^1, \mathbf{H}_\alpha{}^1) = \tfrac{1}{16}B(X_6, X_6) = \tfrac{1}{4}$$

and

$$\mathbf{H}_\alpha = \frac{2\mathbf{H}_\alpha{}^1}{B(\mathbf{H}_\alpha{}^1, \mathbf{H}_\alpha{}^1)} = 2X_6$$

This leads to the result:

$$\begin{aligned} \alpha(\mathbf{H}_\alpha) &= B(\mathbf{H}_\alpha, \mathbf{H}_\alpha{}^1) \\ &= B(2X_6, \tfrac{1}{4}X_6) = \tfrac{1}{2}B(X_6, X_6) \\ &= 2 \text{ as required} \end{aligned} \tag{6.11}$$

Similarly

$$B(X_6, \mathbf{H}_\beta{}^1) = +1$$

so that

$$\mathbf{H}_\beta{}^1 = \tfrac{1}{4}X_6$$

and

$$\mathbf{H}_\beta = 2X_6$$

with

$$\beta(\mathbf{H}_\beta) = 2 \tag{6.12}$$

Also we have:

$$\begin{aligned}\alpha(\mathbf{H}_\beta) &= B(\mathbf{H}_\beta, \mathbf{H}_\alpha^1) = B(2X_6, \tfrac{1}{4}X_6)\\ &= \tfrac{1}{2}B(X_6, X_6) = 2\\ \beta(\mathbf{H}_\alpha) &= B(\mathbf{H}_\alpha, \mathbf{H}_\beta^1)\\ &= B(2X_6, \tfrac{1}{4}X_6) = 2\end{aligned}$$

so that

$$n_{\alpha\beta} = \alpha(\mathbf{H}_\beta) = \frac{2B(\mathbf{H}_\beta^1, \mathbf{H}_\alpha^1)}{B(\mathbf{H}_\beta^1, \mathbf{H}_\beta^1)} = 2$$

$$n_{\beta\alpha} = \beta(\mathbf{H}_\alpha) = \frac{2B(\mathbf{H}_\alpha^1, \mathbf{H}_\beta^1)}{B(\mathbf{H}_\alpha^1, \mathbf{H}_\beta^1)} = 2$$

Then

$$\frac{n_{\alpha\beta}n_{\beta\alpha}}{4} = 1 = \cos^2\phi(\alpha\beta) \text{ or } \cos\phi = +1$$

That is $\phi = 0$ or $180°$.

This agrees with the fact that L_A is one-dimensional in this case of $SO(3, 1)$, so that the two roots $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are along the same real line. (The concept of roots as discussed here and the resulting angle between root vectors, must not be confused with root vector analysis (Pontryagin, 1966; Rowlatt, 1966; Behrends *et al.*, 1962) based on the maximal abelian algebra L^0 of equation (5.1). Here our root vector analysis is based on L_A. We shall discuss the connection between the two approaches elsewhere.)

Returning to the structural and decomposition problem, we find that if we denote by Δ the set of all roots of the algebra L, as computed by the above prescription, then we can write:

$$L = L^0 \oplus \sum_{\alpha \in \Delta} L^\alpha$$

Thus for $sO(3, 1)$ we have

$$sO(3, 1) = L^0 \oplus L^{+1} \oplus L^{-1} \tag{6.13}$$

where L^0, L^{+1} and L^{-1} are given by equations (5.1) and (5.4).

7. *Hyperplanes of L*

Also given now the roots we can deduce information about the algebraic structure of the subset P given in equations (3.1) and (3.5). For this purpose one introduces the notion of positive and negative roots by first defining a set of hyperplanes in the space $\tilde{L}_A$. Generally this is done as follows:

For any given root $\boldsymbol{\alpha}$, we consider all those vectors **H** in space L_A which are such that $\alpha(\mathbf{H}) = 0$. Denoting the set of such vectors by P_α, we have that

$$P_\alpha = \{H \in L_A : \alpha(H) = 0\}$$

We consider all such vectors **H** as lying on a hyperplane denoted by P_α. It is called the hyperplane associated with the root vector $\boldsymbol{\alpha}$. Since

$$\alpha(\mathbf{H}) = \boldsymbol{\alpha}.\mathbf{H}$$

it follows that the hyperplane P_α is orthogonal to the root vector $\boldsymbol{\alpha}$. That is P_α is orthogonal to H_α or to $\mathbf{H}_\alpha{}^1$ in the space L_A. These hyperplanes are found in general by simply looking at the eigenvalue equation such as equation (5.3). In this way, one finds the hyperplanes for each of the other root vectors $\boldsymbol{\beta}, \boldsymbol{\gamma} \ldots$. These hyperplanes $P_\beta, P_\gamma \ldots$ are orthogonal, respectively, to the root vectors $\boldsymbol{\beta}, \boldsymbol{\gamma} \ldots$.

In the case of $SO(3, 1)$ we expect to have two hyperplanes $P_\alpha = P_{(+1)}$ and $P_\beta = P_{(+1)}$, corresponding to our two roots. Since L_A consists of only one operator X_6, the hyperplanes in this case are simply points on the real line. The two points are the origin.

8. *Weyl Group and Weyl Chambers*

Having obtained the hyperplanes, one computes next the set of their associated Weyl chambers and Weyl reflection operators. Abstractly, these concepts are introduced as follows: For each root $\boldsymbol{\alpha}$ we are required to introduce an operator, S_α, with the following properties:

(i) Acting on an arbitrary vector **H** in L_A we should have:

$$S_\alpha \mathbf{H} = \mathbf{H} - \alpha(\mathbf{H})\mathbf{H}_\alpha$$

(ii) Also

$$S_\alpha P_\alpha = P_\alpha$$

(iii) $$(S_\alpha)^2 = I$$

(iv) $$B(\mathbf{H}, \mathbf{H}) = B(S_\alpha \mathbf{H}, S_\alpha \mathbf{H})$$

(v) $B(\mathbf{H}, \mathbf{H}_\alpha) = 0$ for any **H** lying wholly on P_α

(vi) $S_\alpha \mathbf{H}_\alpha = -\mathbf{H}_\alpha$

The computation of such Weyl operators S_α is quite straightforward. Thus consider the case of $SO(3, 1)$.

Since L_A is one-dimensional, the results are rather trivial, but we can still write:

$$S_\alpha = S_{+1}; \qquad S_\beta = S_{+1}$$
$$S_+ X_6 = +X_6 - 2X_6 = -X_6$$
$$(S_+)^2 X_6 = +X_6, \text{ etc.}$$

We get that for $SO(3, 1)$ all such operators consist of S_+, S_- and I. They reduce to a finite group of order 3, known as the Weyl group for $SO(3, 1)$. Here the Weyl group acts in a one-dimensional space spanned by X_6, so that we can write:

$$\text{Weyl Group for } SO(3, 1) = (+1, -1, 1) = (w, w^{-1}, e)$$

Next we consider the Weyl chambers defined as the space between any two hyperplanes in the vector space L_A. These chambers can be computed from the information already available on the hyperplanes. In the case of $SO(3, 1)$ the Weyl chambers consist of the line segments:

$$(0, +\infty) \qquad \text{and} \qquad (-\infty, 0) \tag{8.1}$$

We may denote these chambers by C_+ and C_- respectively. Their relation to the Weyl group is that the elements of the Weyl group have the general property that, acting on roots in one Weyl chamber, they carry them across some hyperplane into another Weyl chamber. The roots in the chamber C_+ are called positive roots while the roots in the chamber C_- are the negative roots. We note, however, that the choice of which roots are positive and which are negative is largely arbitrary. This is particularly obvious when we consider the more general case. However for $SO(3, 1)$ we shall put

$$\begin{aligned} C_+ &= (\alpha, \beta) = (+1, +1) \\ C_- &= (-\alpha, -\beta) = (-1, -1) \end{aligned} \tag{8.2}$$

Then from equation (5.3), one has that the eigenfunctions of the positive roots are Z^+ and W^+. One verifies that Z^+ and W^+ form a nilpotent abelian subalgebra of $SO(3, 1)$, which we may denote by L_N^+. Similarly Z^- and W^- form a nilpotent abelian subalgebra L_N^-.

9. *The Iwasawa Decomposition*

Given now the above detailed information about the structure of the Lie algebra L, one can complete the problem of decomposition of the algebra by appealing to the well-known theorem (Hermann, 1966; Helgason, 1962), according to which the Lie algebra L can uniquely be written in the form:

$$L = L_K \oplus L_A \oplus L_N^+ \tag{9.1}$$

For $sO(3, 1)$ this means:

$$sO(3, 1) = sO(3, R) \oplus X_6 \oplus L_N^+ \tag{9.2}$$

This may be carried over to the group level, leading to the Iwasawa factorisation of the connected analytic group G whose algebra is L. In general if K, A and N^+ stand for the connected analytic subgroups of G which correspond to the Lie algebras L_K, L_A and L_N^+ respectively, the Iwasawa factorisation of G is

$$G = K . A . N^+ \tag{9.3}$$

That is, every element $g \in G$ can be written in general as $g = kan^+$, the order of the factors being also immaterial. For the case of $SO(3, 1)$ discussed here, these analytic subgroups are easily calculated as follows: Let a stand for an arbitrary element of the group A, while n^+ is the arbitrary element of the group L_N^+. Then we have:

$$a = \exp(tX_6) = I + tX_6 + \frac{(tX_6)^2}{2!} + ---\cdots$$

where t is an arbitrary real parameter with $-\infty \leqslant t \leqslant +\infty$. Substituting the matrix representation for X_6 from equation (2.5), we get that

$$a = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh t & \sinh t \\ 0 & 0 & \sinh t & \cosh t \end{pmatrix} \tag{9.4}$$

Next we have that

$$n^+ = \exp(sZ^+ + rW^+) \tag{9.5}$$

where r and s are also real parameters with $-\infty \leqslant r, s \leqslant +\infty$.

From equations (2.5) and (5.2) we get that

$$Z^+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}; \qquad W^+ = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{9.6}$$

Then expanding equation (9.5), and substituting equation (9.6), we obtain that

$$n^+ = \begin{pmatrix} 1 & 0 & -r & r \\ 0 & 1 & -s & s \\ r & s & 1 - \frac{1}{2}(r^2 + s^2) & \frac{1}{2}(r^2 + s^2) \\ r & s & -\frac{1}{2}(r^2 + s^2) & 1 + \frac{1}{2}(r^2 + s^2) \end{pmatrix} \tag{9.7}$$

Finally one evaluates an arbitrary element of the maximal compact subgroup K. In the present case of $SO(3, 1)$, K is the rotation group $SO(3, R)$, whose arbitrary element, in the Euler form, is very well known (Edmonds, 1957) and need not be written down here. In general, one can deduce this arbitrary element $k \in SO(3, R)$ directly from our procedure by writing

$$k = \exp(\theta_1 X_1 + \theta_2 X_2 + \phi X_4) \tag{9.8}$$

where θ_1, θ_2, and ϕ are real bounded parameters, with $0 \leqslant \phi \leqslant \pi$; $0 \leqslant \theta_1$, $\theta_2 \leqslant 2\pi$. Expanding equation (9.8), and substituting our explicit representation given in equation (2.5) we obtain the required form of k. Taking the product of the matrix (9.8) with the matrices in equations (9.4) and (9.7), we finally get an arbitrary element of $SO(3, 1)$ in parameterised Iwasawa form.

Other types of decomposition of $SO(3, 1)$, as discussed for example by Naimark (1964), can be deduced by the procedure followed here. In particular, we can also decompose an arbitrary element of $SO(3, 1)$ in the form

$$g = kak^1$$

where k, k^1 are some two elements of $SO(3,R)$, while $a = a(t) \in A$.

The results for $SL(2, C)$ are in homomorphic correspondence with those of $SO(3, 1)$ and need not be discussed separately. One finds that for $SL(2, C)$ the Iwasawa decomposition at the algebraic level is:

$$sl(2, c) = su(2) \oplus X_6 \oplus L_N^+$$

while at the group level we have:

$$g = uan^+$$

$$n^+ = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$$

$$a = \begin{pmatrix} e^{\frac{1}{2}t} & 0 \\ 0 & e^{-\frac{1}{2}t} \end{pmatrix}$$

$$u = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}; \qquad |\alpha|^2 + |\beta|^2 = 1$$

and $-\infty \leqslant t, r, s, \leqslant +\infty$ as before, and where $\mu = (s - ir)$ = arbitrary complex number. Detailed discussions of this specific case of $Sl(2, C)$ are available in several places (Naimark, 1964; Ruhl, 1970; Gelfand *et al.*, 1963).

10. *Conclusion*

We see how by exploiting various theorems dealing with the properties of Killing forms of Lie algebras one can reduce the problem of analysing the structure of semi-simple Lie algebras and obtaining their factorisations to a series of prescriptions, all of which are amenable to straightforward computations. The parameters which contain the most important information about the structure of a semi-simple Lie algebra may be singled out as follows:

(a) The structure constants.
(b) The Killing forms of the generators.
(c) The eigenvalue spectrum and the eigenfunctions.
(d) The hyperplanes, the Weyl chambers and the Weyl group.

Given these parameters, the structure and factorisation of any semi-simple Lie algebra can be deduced. In order to construct representations of the algebras and the groups, one also needs these structural parameters which are therefore of great importance. Where the group G becomes a physical symmetry group, the structural parameters can acquire some physical meaning.

Elsewhere we shall give further results on the structural parameters of other physically interesting Lie algebras. Applications of the results to specific physical problems will also be considered.

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